On One Uniqueness Theorem for M. Rietz Potentials

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We prove that there exists a nonzero holderian function $f : \mathbb{R} \to \mathbb{R}$ vanishing together with its M. Rietz potential $f * \frac{1}{|x|^{1-\alpha}}$ in all points of some set of positive length. This result improves the one of D. Beliaev and V. Havin [2]

0. Introduction. Let α be a real number, $0 < \alpha < 1$. Let $f : \mathbb{R} \to \mathbb{C}$ be a locally summable function satisfying the following condition:

$$\int_{\mathbb{R}} \frac{|f(x)|dx}{1 + |x|^{1-\alpha}} < +\infty. \tag{1}$$

Let

$$(U_\alpha f)(t) := \int_{\mathbb{R}} \frac{f(x)dx}{|t - x|^{1-\alpha}},\ t \in \mathbb{R}.$$

In case (1) the function $U_\alpha f$ is defined a.e. on $\mathbb{R}$. We call it *the M. Rietz Potential*, and we call f *the density* of this potential. We write $domU_\alpha$ (the domain of U_α) for the set of all locally summable functions satisying (1).

Let $V \in \mathbb{R}$ be a measurable set; we denote its length as $|V|$. The uniqueness theorem mentioned in the title states that *if f satisfies (1) and Hölder's condition with an exponent more than $1 - \alpha$ in some neighboarhood of V, while $|V| > 0$ and*

$$f|_V = U_\alpha|_V = 0, \tag{2}$$

then $f = 0$ a.e. on $\mathbb{R}$.

This theorem follows from a slightly more general "uncertainty principle" proven in [5]. It concerns M. Rietz's potentials of *charges* (not necessarily absolutely continuous with respect to Lebesgue measure) and α's not necessarily from $(0, 1)$; for the history of the problem and its connections with the uniqueness problem for Laplase's equation see also [6] and [2]. Havin [5] posted the following question: is it possible to omit Hölder's condition on f near V in Theorem 1? Moreover, it was still unknown if there exist a nonzero *continuous* function $f \in dom\, U_\alpha$ and a set V of positive length satisfying (2).

In [2], it was shown that the answer to the last question is affirmative. However, the function f constructed in [2], being continuous, does not satisfy any Hölder's condition.

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In this paper, we build a nonzero *hölderian* function $f \in domU_\alpha$ vanishing with its M. Rietz potential $U_\alpha f$ on some set of positive length.

As well as in [2], we build the desired function using the techniques of "correction" which were proposed by D. Menshov and applied to the problems of potential theory in [1], [3], [4], [7]. Our progress (as compared to [2]) is based on improving the correction process using elementary probabilistic techniques (see Lemma 5[2]). We also mention that we deal with *complex* (not only real-valued!) densities f. This detail seems unessential, but it surprisingly simplifies the construction of a desired (*real-valued!*) density f, even in case [2], when the goal was just to find *continuous* (not necessarily hölderian) f.

The main result of this paper is the following theorem.

Theorem 1 *There exist a nonzero function $f \in dom\, U_\alpha$, a set $V \subset \mathbb{R}$ of positive length and a positive number r, such that f and $U_\alpha f$ are identically zeroes on V and f satisfies Hölder's condition with exponent r.*

Remark. The function f we build is complex-valued. In order to get a real-valued density, one should just take its real or imaginary part; at least one of them will not be an identical zero.

I am grateful to V. Havin for introducing me to the problem and for useful discussions.

1. Operator W_α. We will need an operator that is in some sense inverse to U_α. Let for $g \in C_0^\infty(\mathbb{R})$

$$(V_\alpha g)(t) := \frac{1}{\alpha} \int_{\mathbb{R}} g(x) \frac{sgn(t-x)}{|t-x|^\alpha} dx,$$

$$(W_\alpha g)(t) = ((V_\alpha g)(t))', \quad t \in \mathbb{R}.$$

Let g be a function defined on $\mathbb{R}$, $\lambda > 0$, $\varepsilon > 0$. Denote $(C_\lambda g)(x) = g(\lambda x)$, $g_\varepsilon = \frac{1}{\varepsilon} C_{1/\varepsilon} g$. The next lemma states some properties of W.

Lemma 1 *1) $W_\alpha(C_0^\infty(\mathbb{R})) \subset \mathrm{dom} U_\alpha$;*

2) $U_\alpha W_\alpha g = cg, g \in (C_0^\infty(\mathbb{R}))$;

3) $W_\alpha C_\lambda = \lambda^\alpha C_\lambda W_\alpha$;

[2]I am grateful to S. Smirnov for useful discussions concerning this lemma.

4) $\alpha W_\alpha g = -g' * \frac{sgn x}{|x|^\alpha}$ *(we use $*$ for the convolution on $\mathbb{R}$);*

5) $(W_\alpha g)(t) = (g * |x|^{-\beta})(t), g \in \mathrm{dom} W_\alpha, t \notin \operatorname{supp} g.$

Hereinafter $\beta = \alpha + 1$. All statements of this lemma are well-known (2) or obvious (1,3,4,5). See, for example, [2, page 226].

We use the following notation: $I = (-\frac{1}{2}, \frac{1}{2})$; if Q is a bounded interval, then c_Q is its center.

We write $\phi(t)$ for "the finitizator": $\phi \in C^\infty$, $\operatorname{supp} \phi \subset I$, $\int_I \phi = 1$, $\phi \geq 0$.

In the proof we shall fix positive numbers p and λ. For a function $h : \mathbb{R} \to \mathbb{C}$ we introduce its "embedding to the interval Q" $h_Q(t) := h(\frac{t-c_Q}{|Q|\lambda})$, $t \in \mathbb{R}$. Finally, let $M_Q(h) := (\frac{1}{|Q|} \int_Q |h|^p)^{1/p}$.

2. Main Lemma. The mail tool for the proof of Theorem 1 will be Lemma 4. First, we prove auxiliary lemmas 2 and 3. We state the existence of functions with certain concrete numerical properties. These functions will serve as "building blocks" for our construction. The following proposition is principal for us: for some constant $B < 0$ and some $p > 0$ there exists a function $h \in C_0^\infty$ with arbitrarily small support satisfying $\int_\mathbb{R} (|1 - W_\alpha h|^p - 1) < B$. The meaning of this fact is that it allows us to control both the length of support of h (future correcting term) and its influence on the "amount" of potential W_α of the function we are correcting. We shall rearrange this statement for it to be convenient for our purposes.

The function h will be made from "the finitizator" ϕ by means of an appropriate scaling.

Let for $\varepsilon > 0$, $t \in \mathbb{R}$

$$F^{[\varepsilon]}(t) := (W_\alpha \phi_\varepsilon)(t), \quad F^{[0]}(t) := |t|^{-\beta}.$$

Lemma 2 *If a positive number p is sufficiently small, then*

$$J(p) := \int_\mathbb{R} (|1 - F^{[0]}|^p - 1) dx < 0$$

Proof. Let $L := \int_\mathbb{R} \log |1 - F^{[0]}|$. Then, as $(a^p - 1)/p$ is monotone in p for any $a > 0$ and converges to $\log a$ as $p \to 0$, we have $\lim_{p \searrow 0} \frac{J(p)}{p} = L$ (note that $||1 - F^{[0]}(t)|^p - 1| \leq c|t|^\beta$, when $|t|$ is large, and if $p < 1/\beta$, then the integral $J(p)$ converges in zero as well). But L can be computed exactly:

$L = 2\pi \cot \frac{\pi}{\beta} < 0$. One can find the computation, for example, in [2, page 233].

From now on the number p found in the previous lemma will be fixed.

In the next lemma we pass from $F^{[0]}$ (the potenial W_α of the delta-function) to the potential of some concrete function ϕ_ε. We also introduce a "small complex rotation": multiply ϕ_ε by $e^{i\theta}$ with small θ. This leads to some technical simplifications in the future.

Lemma 3 *There exist numbers $B < 0$, $\theta_0 > 0$ and $\varepsilon_0 > 0$, such that if $0 \le \theta \le \theta_0$ and $0 \le \varepsilon < \varepsilon_0$, then*

$$J(\varepsilon, \theta) := \int_{\mathbb{R}} (|1 - e^{i\theta} F^{[\varepsilon]}|^p - 1) < B.$$

Proof. Using the homogeneity property of W_α (point 3 of Lemma 1), we get

$$|F^{[\varepsilon]}(t)| \le C(\alpha) \min(\frac{1}{\varepsilon^\beta}, \frac{1}{|t|^\beta}), \quad t \in \mathbb{R}. \tag{3}$$

It is clear that $F^{[\varepsilon]}$ converges to $F^{[0]}$ pointwise as $\varepsilon \to 0$. It follows from Lebesgue dominated convergence theorem that

$$\lim_{\varepsilon \searrow 0} J(\varepsilon, 0) = J(p).$$

Choose $B < 0$ to satisfy $J(\varepsilon, 0) < 2B$ for $\varepsilon \in (0, \varepsilon_0)$. We are to prove that $J(\varepsilon, \theta) \xrightarrow{\theta \to 0} J(\varepsilon, 0)$ uniformly in ε. Indeed,

$$|J(\varepsilon,\theta) - J(\varepsilon, 0)| = \left| \int_{\mathbb{R}} |1 - e^{i\theta} F^{[\varepsilon]}|^p - |1 - F^{[\varepsilon]}|^p \right| \le \left| \int_{|x| > \theta^{-p/2}} \right| + \left| \int_{|x| < \theta^{-p/2}} \right| =: J_1 + J_2.$$

We have

$$J_1 = \left| \int_{|x| > \theta^{-p/2}} |1 - e^{i\theta} F^{[\varepsilon]}|^p - |1 - F^{[\varepsilon]}|^p \right| \le 2C \left| \int_{|x| > \theta^{-p/2}} |t|^{-\beta} \right| \xrightarrow{\theta \to 0} 0,$$

as the integral converges in infinity (and does not depend on ε). Then, using the inequality $|a^p - b^p| \le |a - b|^p$ for $a > 0$, $b > 0$ and $p \in (0, 1)$, we get

$$J_2 = \left| \int_{|x| < \theta^{-p/2}} |1 - e^{i\theta} F^{[\varepsilon]}|^p - |1 - F^{[\varepsilon]}|^p \right| =$$

$$= \left| \int_{|x|<\theta^{-p/2}} |e^{-i\theta} - F^{[\varepsilon]}|^p - |1 - F^{[\varepsilon]}|^p \right| \leq$$

$$\leq \int_{|x|<\theta^{-p/2}} |e^{-i\theta} - 1|^p \leq \int_{|x|<\theta^{-p/2}} \theta^p \xrightarrow{\theta\to 0} 0.$$

So, $J_1 + J_2 \xrightarrow{\theta\to 0} 0$ uniformly in $\varepsilon \in (0, \varepsilon_0)$, and the lemma is proved.

We are now ready to formulate Lemma 4 – the main tool of further construction. We replace the whole line (from Lemma 3) by a bounded interval, the constant 1 by an arbitrary function with a small oscillation, and finally we bring in a (small) positive parameter λ. This one is responsible for smallness of the potential W_α of the correcting term far away from the interval we correct on. Denote $\gamma(\lambda) := (1 + B\lambda/2)^{1/p}$, where B is a constant from Lemma 3. Notice that $0 < \gamma(\lambda) < 1$ for any sufficiently small positive λ.

If f is a function defined on some interval Q, we define $osc_Q f := \sup\limits_{x,y\in Q} (|f(x) - f(y)|)$ (*the oscillation* of f on Q).

Lemma 4 *There exist numbers $\theta > 0$, $\lambda_0 > 0$, $\varepsilon_0 > 0$, such that for any positive $\lambda < \lambda_0$ one can find a number $\kappa > 0$ with the following property: if $0 < \varepsilon < \varepsilon_0$, Q is any bounded interval and a continuous complex-valued function h satisfies*

$$osc_Q h \leq \kappa |h(c_Q)|, \tag{4}$$

then

1) $M_Q(h - h(c_Q)e^{i\theta}F_Q^{[\varepsilon]}) \leq \gamma(\lambda)|h(c_Q)|$;

2) $\frac{\theta}{2}|h(t)| \leq |h(t) - h(c_Q)e^{i\theta}F_Q^{[\varepsilon]}(t)| \leq \frac{C}{\varepsilon^\beta}|h(t)|, \quad t \in Q.$

Recall that $F_Q^{[\varepsilon]}(t) = F^{[\varepsilon]}(\frac{t-c_Q}{|Q|\lambda})$.

Proof. Take ε_0 and θ from Lemma 3. First we get an estimate in point 2:

$$|h(t) - h(c_Q)e^{i\theta}F_Q^{[\varepsilon]}| \geq |h(c_Q)||1 - e^{i\theta}F_Q^{[\varepsilon]}(t)| - |h(t) - h(c_Q)| \geq |h(c_Q)|\frac{3\theta}{4} - \kappa|h(c_Q)| =$$

$$= |h(c_Q)|(\frac{3\theta}{4} - \kappa) \geq |h(t)|\frac{1}{1+\kappa}(\frac{3\theta}{4} - \kappa) \geq |h(t)|\frac{\theta}{2}$$

for any sufficiently small κ. We have used an elementary inequality $dist(1, \{re^{i\theta} : r \in \mathbb{R}\}) = \sin\theta \geq \frac{3\theta}{4}$, if $\theta > 0$ is small.

The right-hand inequality in the point 2 follows clearly from (3).

Now we prove point 1:

$$(M_Q(h - h(c_Q)e^{i\theta}F_Q^{[\varepsilon]}))^p \leq (M_Q(h - h(c_Q)))^p + |h(c_Q)|^p(M_Q(1 - e^{i\theta}F_Q^{[\varepsilon]}))^p \leq$$

$$\leq (osc_Q h)^p + |h(c_Q)|^p(M_{\lambda^{-1}I}(1 - e^{i\theta}F^{[\varepsilon]}))^p.$$

Notice that

$$(M_{\lambda^{-1}I}(1 - e^{i\theta}F^{[\varepsilon]}))^p = 1 + \lambda \int_{\lambda^{-1}I}(|1 - e^{i\theta}F^{[\varepsilon]}|^p - 1) =$$

$$= 1 + \lambda \int_{\mathbb{R}}(|1 - e^{i\theta}F^{[\varepsilon]}|^p - 1) - \lambda \int_{\mathbb{R}\setminus\lambda^{-1}I}(|1 - e^{i\theta}F^{[\varepsilon]}|^p - 1).$$

Taking (3) into account, we get

$$\left|\lambda \int_{\mathbb{R}\setminus\lambda^{-1}I}(|1 - e^{i\theta}F^{[\varepsilon]}|^p - 1)\right| \leq C\lambda^{\alpha+1} = o(\lambda), \lambda \longrightarrow 0,$$

so, if λ is sufficiently small, we have $M_{\lambda^{-1}I}(1 - e^{i\theta}F^{[\varepsilon]}) \leq 1 + B\lambda/2 < 1$. Therefore

$$M_Q(h - h(c_Q)e^{i\theta}F^{[\varepsilon]})^p \leq |h(c_Q)|^p(1 + \left(\frac{osc_Q(h)}{h(c_Q)}\right)^p + 2B\lambda/3) \leq |h(c_Q)|^p(1 + \kappa^p + 2B\lambda/3)).$$

If κ is sufficiently small, then $(1 + \kappa^p + 2B\lambda/3)) < 1 + B\lambda/2$. The lemma is proved.

Remark 1. Careful examination of the proof shows that one can take κ equal to $\min((\frac{|B|\lambda}{2})^{1/p}, \frac{\theta}{8})$, if θ is not too large.

Remark 2. It is the left-hand inequality in the first point for what we pass to complex-valued functions. One cannot obtain such an estimate for real-valued functions.

Remark 3. By this moment we have fixed parameters p and θ. In what follows, constants that we regard as depending on α may also depend on these parameters. Later we shall fix an appropriate λ and, thus, κ and γ.

3. General idea of the construction. Now we describe the plan of construction of f and V (see statement of Theorem 1). We shall build a sequence of functions g_n, $g_n = g_{n-1} - r_{n-1}$, and a decreasing sequence of sets $V_n \subset I$ with the following properties:

1) A nonzero function g_1 belongs to C_0^∞, and *supp* $g_1 \subset \mathbb{R}\backslash I$;

2) $r_k \in C_0^\infty$, and *supp* $r_k \subset I$ for all $k \in \mathbb{N}$;

3) $\sum\limits_{k=1}^{\infty} |supp\ r_k| < \frac{1}{4}$;

4) $|\bigcap\limits_{k=1}^{\infty} V_n| > \frac{3}{4}$;

5) $\int_{V_n} |f_n|^p \xrightarrow{n\to\infty} 0$; here $f_n := W_\alpha g_n$ and p is the positive number fixed in the previous section

6) Sequences g_n and f_n converge uniformly on $\mathbb{R}$ to some continuous functions g and f correspondingly, and $g = U_\alpha f$.

Let $V := \bigcap\limits_{k=1}^{\infty} V_n$, $V' := \{x \in I :\ g(x) = 0\}$. It follows from 1, 3 and 4 that $|V'| > \frac{3}{4}$ and $|V| > \frac{3}{4}$. Therefore $|V \cap V'| > \frac{1}{2}$. It follows from 5 and 6 that $f|_V = 0$. Finally, using 2, we conclude that $g|_{\mathbb{R}\backslash I} = g_1|_{\mathbb{R}\backslash I}$, and, thus, the function g is not identically 0. Hence the set $V \cap V'$ and the function f satisfy all conditions of Theorem 1, except (may be) Hölder's condition.

Now we describe more precisely the structure of sets V_n and correcting terms r_k. Bring in a *sequence of positive numbers* $\{\delta_n\}_1^\infty$. It will have the following properties: $\delta_1 = 1, \frac{\delta_n}{\delta_{n+1}} \in \mathbb{N}$. We denote the partition of the interval I to intervals of length δ_n as H_n. The set V_n will be obtained as the union $\bigcup_{Q\in G_n} Q$, where G_n is some subset of H_n. Roughly speaking, the set G_n consists of all intervals on which we haven't finish correction yet, in particular, for all $k > n$ there holds *supp* $r_k \subset V_n$.

Let us fix a sequence of positive numbers $\{\varepsilon_n\}_{n=1}^\infty$, such that $\sum\limits_{n=1}^{\infty} \varepsilon_n < \frac{1}{4}$, and, in addition, ε_n decay not very fast: $\varepsilon_n^{-1} = O(n^m)$ for some $m > 0$. It will be responsible for the length of supports of r_n: there will hold an estimate $|supp\ r_n| \le \varepsilon_n$ for all n.

We also demand *supp* $g_1 \in (\frac{1}{2}, \frac{3}{2})$, and, moreover, $\forall t \in I f_1(t) \neq 0$. One can take, for example, $g_1 := \phi(x-1)$.

Then, we choose some subset $G_n^g \subset G_n$ and let

$$r_n := \sum_{Q\in G_{n+1}^g} (\lambda\delta_{n+1})^\alpha f_n(c_Q)(\phi_{\varepsilon_n})_Q e^{i\theta}. \tag{5}$$

So,

$$W_\alpha r_n = \sum_{Q\in G^g_{n+1}} f_n(c_Q) F^{[\varepsilon_n]}_Q e^{i\theta}.$$

Note that in such a definition condition 3 will be satisfied by choosing the sequence ε_n as described abowe.

The idea is that if δ_{n+1} is sufficiently small, then on each interval $Q \in G^g_{n+1}$ the oscillation of f_n is small (there holds estimate (4)), and one can apply Lemma 4 with f_n as h. Its result, together with an observation that functions $F^{[\varepsilon]}_Q$ decay sufficiently fast far away from Q, allow us, using the notation $V^g_n := \bigcup\limits_{Q\in G^g_n} Q$, to prove an estimate

$$\int_{V^g_{n+1}} |f_{n+1}|^p \le \eta \int_{V^g_n} |f_n|^p \tag{6}$$

with some $\eta \in (0,1)$. If one chooses G^g_n appropriately (if on each step they occupy a large part of G_n), this leads to an estimate of integral over the whole set V_n:

$$\int_{V_n} |f_n|^p = O(\eta^{\frac{n}{2}}). \tag{7}$$

Hence condition 5 will be obtained.

Remark 1. The choice of G_n (decreasing V_n on each step) allows us to make functions f_n converge not only in the sense of $L^p(I)$, but uniformly, in partiular, we get an estimate $|f_n(c_Q)| = O(\eta'^n)$, $Q \in G_n$, where $\eta' \in (0,1)$.

Remark 2. If we did not worry about the control over modulus of continuity, we could take $G^g_n := G_n$. Then (7) automatically follows from (6), and the whole construction becomes more simple. Unfortunately, in order to get Hölder's condition, one should pick out the set G^g_n on each step – this is a set of intervals where the oscillation of f_n is especially small – and make the correction only there.

4. Remarks on the estimate of modulus of continuity. In this section, we explain (not quite rigorously) what does estimates of modulus of continuity of f depend on.

We use the following simple fact: *if a sequence of functions h_n converges on $\mathbb{R}$ to the function h, whereas $|h_n - h| \le C_1\eta_1^n$, and $|h'_n| \le C_2 R^n$ (here $\eta_1 \in (0,1)$, $R > 1$), then h satisfies Hölder's condition with an exponent* $\log\eta_1/\log\frac{\eta_1}{R}$.

The condition $|f_n - f| \le C_1\eta_1^n$, will follow from Remark 1 at the end of the previous section (and, in fact, from estimate (7)). When one estimates the derivative $f_n'(t)$ of the function f_n, the main role is played by the last added term $W_\alpha r_{n-1}$, or, more precisely, the building block $f_{n-1}(c_{Q_t})F_{Q_t}^{[\varepsilon_{n-1}]}e^{i\theta}$, where the interval $Q_t \in H_n$ is defined by the statement $t \in Q_t$. From the homogeneity properties of W_α one can get (for $Q \in H_n$) an estimate

$$|(F_Q^{[\varepsilon_{n-1}]})'(t)| \le \frac{c}{\varepsilon^{-\beta-1}\lambda\delta_n}\ t \in \mathbb{R} \tag{8}$$

Thus, one can get (say, on V_n) an estimate

$$|f_n'| \le \frac{Cf_n(c_{Q_t})}{\delta_n\varepsilon_{n-1}^{\beta+1}}, \tag{9}$$

and so, everywhere,

$$|f_n'| \le \frac{C\eta'^n}{\delta_n\varepsilon_{n-1}^{\beta+1}}. \tag{10}$$

This means that the obtained function f would be hölderian if numbers δ_n^{-1} did not grow faster than some geometric series, in other words, if every time we divided the interval $Q \in H_n$ into the same number of parts. On the other hand, the exponent $\log\eta_1/\log\frac{\eta_1}{R}$ tends to zero as $R \to \infty$, hence it is clear that if $\frac{\delta_n}{\delta_{n+1}} \to \infty$ as $n \to \infty$, then we are unable to prove Hölder's condition with any exponent.

It is clear, however, that if in order to define δ_{n+1} we use a natural estimate (9) (recall what the necessity to choose small δ_{n+1} is due to: we need to estimate the oscillation of f_n in order to use Lemma 4 – condition (4)), then because of the increasing multiplier $\varepsilon_{n-1}^{-\beta-1}$ we should take δ_{n+1}/δ_n tending to zero to make (4) hold. Therefore we need finer esimates of modulus of continuity of f_n, holding, however, not on the whole set V_{n+1}, but on some "good" part V_{n+1}^g of it.

Note that the building block $F_Q^{[\varepsilon_n]}$ and its derivative $(F_Q^{[\varepsilon_n]})'$ are large in modulus (as $\varepsilon_n^{-\beta}$ and $\delta_{n+1}^{-1}\varepsilon_n^{-\beta-1}$ correspondingly) only near the center of Q; if we consider them outside the interval of the length $\tau|Q|$ and with the same center with Q, where $0 < \tau < 1$, then we get estimates $|F_Q^{[\varepsilon_n]}| \le C$ and $|(F_Q^{[\varepsilon_n]})'| \le C\delta_{n+1}^{-1}$.

One has an idea - exclude this "bad" central part of Q and correct on the remaining part only. Unfortunately, if we drop it forever, this will mean that

on each step one eliminates from V_n a subset of length $\tau|V_n|$, and this makes $\cap_{n\in\mathbb{N}}V_n$ have zero length.

Therefore, for each interval from H_n we bring in a system of its "bad" subsets, and on each of them we shall "make a pause" - not correct during the next few steps, until the partition H_{n+k} becomes so fine that the estimates of f_n and its derivative become satisfactory (requirements to this estimates becomes weaker if n grows). The the pause duration depends on the distance of the corresponding subset from the center of the interval, i.e. on how "bad" f_n is on this subset. According to this, G_{n+1} is divided into two parts: G^g_{n+1}, where the correction is made on this step, and G^d_{n+1}, where we do not do anything for the time being.[3]

After that, we shall estimate an amount of intervals from G_n such that have more than one half of their "ancestors" from $G_k,\ k=0,1,\dots n-1,$, belong to G^d_k. It turns out that there are only a few of them (if G^d_k is a small part of G_k for each k), and we drop them out. For the rest, we prove an estimate like (7), using Lemma 4 and fast decay of $F_Q^{[\varepsilon_n]}$ away from Q.

5. Definition of sets G^g_n. To complete the construction, we should determine the sequence δ_n and sets G_n and G^g_n. We need a number of estimates depending on how one picks out "good" subsets G^g_n from G_n, but not on the way to choose G_n themselves. We prove these estimates in sections 6, 7 and 8. Later, in section 9, we will define the way to choose G_n.

Bring in a positive parameter δ, such that $\delta^{-1}\in\mathbb{N}$, and let $\delta_n:=\delta^n$.

Then, bring in a parameter $\tau\in(0,1)$. Let $\tau^{-1}\in\mathbb{N}$, and, moreover, $\tau/\delta\in\mathbb{N}$. We use the following notation: if $a>0$ and Q is a bounded interval, then $Q[a]:=Q\backslash Q'$, where Q' is an interval of the length $a|Q|$ and with the same center as Q.

We are now ready to define the set G^g_{n+1}. An interval $Q\in G_{n+1}$ belongs to G^g_{n+1}, iff for any $k=0,1,\dots,n-1$ there holds an implication $Q\subset Q'\in G^g_{n-k}\Rightarrow Q\subset Q'[\tau^{k+1}]$. In other words, if in the $(n-k)$-th step we have made a correction[4] on the interval Q', then on the next step the correction is forbidden on the set $Q'\backslash Q'[\tau]$, on the $(n-k+2)$-th step it is forbidden on the set $Q'\backslash Q'[\tau^2]$, and so on. It follows from the condition $\tau/\delta\in\mathbb{N}$ that the interval $Q\in G_{n+1},\ Q\subset Q'\in G^g_{n-k}$ either lies in $Q'[\tau^{k+1}]$ or does not intersect it.

[3]Top indices g and d are the first letters of "go" and "delay"

[4]Recall that it means that Q' belongs to the set of indices of summation in the definition (5) of the corresponding correcting term r_{n-k-1}

In fact, $Q'[\tau^{k+1}]\backslash Q'[\tau^k]$ are the very "bad" subsets of Q'; on the k-th of them the "length of pause" is k steps.

Let us make some simple, but important observations. It is easy to see that if Q is a bounded interval and $dist(t, c_Q) > 3\lambda\varepsilon_n|Q|$, then for any $n \in \mathbb{N}$

$$|F_Q^{[\varepsilon_n]}(t)| \le C(\alpha)\frac{(\lambda|Q|)^\beta}{|t-c_Q|^\beta}, \tag{11}$$

and, besides,

$$|(F_Q^{[\varepsilon_n]})'(t)| \le C(\alpha)\frac{(\lambda|Q|)^\beta}{|t-c_Q|^{\beta+1}}. \tag{12}$$

This means that for all $k \in \mathbb{N}$ and for $t \in Q[\tau^k]$

$$|F_Q^{[\varepsilon_n]}(t)| \le C_1(\alpha)\frac{\lambda^\beta}{\tau^{k\beta}}, \tag{13}$$

$$|(F_Q^{[\varepsilon_n]})'(t)| \le C_1(\alpha)\frac{\lambda^\beta}{|Q|\tau^{k(\beta+1)}}. \tag{14}$$

Indeed, for $\tau^k/2 > 3\lambda\varepsilon_n$ these estimates coincide with the previous ones, and for $\tau^k/2 > 3\lambda\varepsilon_n$ we can use estimates

$$|F_Q^{[\varepsilon_n]}(t)| \le \frac{C(\alpha)}{\varepsilon_n^\beta}$$

and

$$|(F_Q^{[\varepsilon_n]})'(t)| \le \frac{C(\alpha)}{|Q|\lambda\varepsilon_n^\beta}, \quad t \in \mathbb{R}.$$

We can improve the right-hand inequality in point 2 of Lemma 4 for $t \in Q[\tau^k]$. Indeed, applying (13) instead of (3), we get

$$|h(t) - h(c_Q)e^{i\theta}F_Q^{[\varepsilon]}(t)| \le \frac{C(\alpha)\lambda^\beta}{\tau^{k\beta}}|h(t)|, \quad t \in Q[\tau^k]. \tag{15}$$

Let $t \in I$. Denote as Q_t^n an element of H_n defined by the condition $t \in Q_t^n$. Let $D_n^k(t) := \#\{l = 1, \dots, n : t \in Q_t^l\backslash Q_t^l[\tau^k]\}$. For $t \in V_n$ let $\widetilde{D}_n(t) := \#\{l = 1, \dots, n : Q_t^l \in G_l^d\}$. We know that if $Q_t^l \in G_l^d$, then for some $k_l \in \mathbb{N}$ there holds an inclusion $Q_t^l \subset Q_t^{l-k_l}\backslash Q_t^{l-k_l}[\tau^{k_l}]$. Moreover, for $l_1 \ne l_2$ either $k_{l_1} \ne k_{l_2}$, or $l_1 - k_{l_1} \ne l_2 - k_{l_2}$. So, with each natural $l \le n$, such that $Q_t^l \in G_l^d$, we can associate a pair $(l-k_l, k_l)$, such that $Q_t^l \subset Q_t^{l-k_l}\backslash Q_t^{l-k_l}[\tau^{k_l}]$, and such

a mapping will be injective. Hence $\widetilde{D}_n(t) \le \sum\limits_{k\in\mathbb{N}} D_n^k(t) =: D_n(t)$. The next section is devoted to estimates of lengths of sets $E_n := \{t \in I : D_n(t) \ge n/2\}$.

6. Estimates of lengths of sets E_n.

Lemma 5 . *For τ sufficiently small there holds an inequality*

$$|E_n| \le \frac{C(\tau)}{n^2}, \tag{16}$$

where $C(\tau)$ tends to zero as $\tau \to 0$.

Proof. Let

$$\xi'^{(k)}_i := \sum_{Q\in H_i} \chi_{Q\setminus Q[\tau^k]};$$

$$\xi^{(k)}_i := \xi'^{(k)}_i - \tau^k.$$

Then, considering $\xi^{(k)}_i$ as random variables on the probabilistic space I with the measure dx, we have $E\xi^{(k)}_i = 0$. We can describe sets E_n in terms of functions $\xi^{(k)}_i$ as follows:

$$x \in E_n \Leftrightarrow \sum_{k=1}^{n-1}\sum_{i=1}^{n-k} \xi'^{(k)}_i(x) \ge \frac{n}{2}$$

Hence we are to estimate probabilities of the event that sums of random variables $\sum_{i=1}^{n-k} \xi'^{(k)}_i$ are large. It is easy to see that random variables $\xi'^{(1)}_i, \quad i = 1,2,\ldots$ are independent: it follows from the fact that τ/δ is an integer. Unfortunately, for $k > 1$ one cannot say the same thing about $\xi'^{(k)}_i, \quad i = 1,2,\ldots$, because τ^k/δ is not necessarily an integer. But, for $k > 1$ variables $\xi'^{(k)}_i, \quad i = 1,2,\ldots$ are still in some sense "almost independent", and we use it.

Note that if $j \ge i + k$, then $\xi^{(k)}_i$ is a constant on each interval from H_j (it follows from inclusion $\tau^k/\delta^k \in \mathbb{N}$). Hence we can made the following observation:

1) If $i_1 + k \le i_2 \le i_3 \le i_4$, then $E(\xi^{(k)}_{i_1}\xi^{(k)}_{i_2}\xi^{(k)}_{i_3}\xi^{(k)}_{i_4}) = 0$ (as the function $\xi^{(k)}_{i_2}\xi^{(k)}_{i_3}\xi^{(k)}_{i_4}$ is periodic with a period equal to δ^{i_1+k}, and $\xi^{(k)}_{i_1}$ is constant on each interval from H_{i_1+k}).

2) If $i_1 \le i_2 \le i_3 \le i_4 - k$, then $E(\xi_{i_1}^{(k)}\xi_{i_2}^{(k)}\xi_{i_3}^{(k)}\xi_{i_4}^{(k)}) = 0$ (as the function $\xi_{i_4}^{(k)}$ is periodic with a period equal to δ^{i_3+k}, and $\xi_{i_1}^{(k)}\xi_{i_2}^{(k)}\xi_{i_3}^{(k)}$ is constant on each interval from H_{i_3+k}).

Now we write

$$P(|\sum_{i=1}^{n}\xi_i^{(k)}| > \varepsilon) \le \frac{E(\xi_1^{(k)}+\ldots+\xi_n^{(k)})^4}{\varepsilon^4} =$$

$$= \frac{\sum\limits_{(i_1,i_2,i_3,i_4)\in\{1,\ldots,n\}^4} E(\xi_{i_1}^{(k)}\xi_{i_2}^{(k)}\xi_{i_3}^{(k)}\xi_{i_4}^{(k)})}{\varepsilon^4}. \tag{17}$$

First note that $E(\xi_{i_1}^{(k)}\xi_{i_2}^{(k)}\xi_{i_3}^{(k)}\xi_{i_4}^{(k)}) \le E|\xi_{i_1}^{(k)}| = 2\tau^k(1-\tau^k) \le 2\tau^k$ (the first inequality follows from the inequality $|\xi_i^{(k)}| < 1$ for all i and k). Second, if (j_1, j_2, j_3, j_4) is a non-decreasing permutation of the numbers (i_1, i_2, i_3, i_4), then a term $E(\xi_{i_1}^{(k)}\xi_{i_2}^{(k)}\xi_{i_3}^{(k)}\xi_{i_4}^{(k)})$ may differ from zero only if $j_2 - j_1 < k$ and $j_4 - j_3 < k$ (by the above observation). But the number of such fours (j_1, j_2, j_3, j_4) does not exceed $\frac{n(n-1)}{2}k^2$. Hence, the number of nonzero terms in the numerator in (17) does not exceed $4!\frac{n(n-1)}{2}k^2$. Therefore

$$P(|\sum_{i=1}^{n}\xi_i^{(k)}| > \varepsilon) \le \frac{Cn^2k^2\tau^k}{\varepsilon^4}. \tag{18}$$

Let $E_n^k := \{t \in I : D_n^k(t) \ge n(\sqrt[8]{(4\tau)^k} + \tau^k)\}$. If τ is so small that $\sum_1^\infty \sqrt[8]{(4\tau)^k} + \tau^k < \frac{1}{2}$, then $|E_n| \le \sum_{k=0}^\infty |E_n^k|$. Finally, note that $D_n^k(t) = \sum_{i=1}^n {\xi'}_i^{(k)}(t) = \sum_{i=1}^n \xi_i^{(k)}(t) + n\tau^k$, so, using (18), we get

$$|E_n| \le \sum_{k=1}^{\infty}|E_n^k| \le \sum_{k=1}^{\infty}\frac{Ck^2\tau^{\frac{k}{2}}}{n^2 2^k}.$$

The lemma follows from this estimate.

7. Estimates of f_n. In this section, we prove some estimates for f_n's, main of them are estimate (24), which allows us to apply Lemma 4, and estimate (20), which shows that for λ sufficiently small the terms $f_n(c_{Q'})F_{Q'}^{[\varepsilon_n]}e^{i\theta}$, corresponding to $Q' \neq Q$, do not change the situation on Q essentially. All the estimates we prove do not depend on the choice of G_n, but depend on how

we pick out subsets G_n^g (namely, we use estimates (13) and (14)). The sets G_n, as mentioned above, will be defined later. Let

$$T_{n+1}(t) := \sum_{Q \in G_{n+1}^g, Q \neq Q_t^{n+1}} f_n(c_Q) F_Q^{[\varepsilon_n]} e^{i\theta}. \tag{19}$$

Recall that an interval $Q_t^{n+1} \in H_{n+1}$ is defined by condition $t \in Q_t^{n+1}$.

Lemma 6 *There exists a positive number $\rho = \rho(\alpha)$, such that for $\lambda > 0$ sufficiently small and for $\delta = \delta(\lambda) > 0$ sufficiently small there holds the following:*

1) for all $t \in I$

$$|T_{n+1}(t)| \leq \frac{c(\alpha)\lambda^\beta |f_n(t)|}{\rho}; \tag{20}$$

2) if $k \leq n$, $x \in V_{n+1}^g$, $y \in I$ and $|x-y| \leq \delta^k$, then

$$|f_n(x)| \leq \frac{|f_n(y)|}{\rho^{n-k+1}}; \tag{21}$$

3) for all $t \in I$, there holds an estimate

$$|T'_{n+1}(t)| \leq \frac{c(\alpha)\lambda^\beta |f_n(t)|}{\delta^{n+1}\rho}; \tag{22}$$

4) for all $t \in V_{n+1}^g$, there holds an estimate

$$|f'_n(t)| \leq \frac{c_1(\alpha)\lambda^\beta |f_n(t)|}{\delta^n}; \tag{23}$$

5) for all $Q \in G_{n+1}^g$, there holds an estimate

$$osc_Q f_n \leq \kappa |f_n(c_Q)|. \tag{24}$$

Proof. Inequality (24) clearly follows from (23), if we take δ sufficiently small (an interval Q in (24) is of the length δ^{n+1}). We derive (20) and (22) from (21), which, in turn, follows from (20) and (24) for preceding n. Finally, (23) follows from (24) and (22) for preceding n.

The base of induction – (24) and (21) for $n = 1$ – is provided by the condition $f_1(t) \neq 0,\ t \in I$ (see Section 3) and the choice of ρ and δ sufficiently small.

Derive (20) and (22) from (21). Fix $t \in I$. Denote as G_ϵ the set of all intervals $Q' \in H_{n+1}$ satisfying the property $dist\,(c_{Q'}, Q_t^{n+1}) \geq \epsilon$. Denote

$$\sigma_\epsilon(t) := \sum_{Q' \in G_\epsilon} |F_{Q'}^{[\varepsilon_n]}(t)|,$$

$$\sigma_\epsilon^*(t) := \sum_{Q' \in G_\epsilon} |(F_{Q'}^{[\varepsilon_n]})'(t)|.$$

We need estimates

$$\sigma_\epsilon(t) \leq c(\alpha)\lambda^\beta \left(\frac{\delta^{n+1}}{\epsilon}\right)^\alpha \tag{25}$$

and

$$\sigma_\epsilon^*(t) \leq c(\alpha)\lambda^\beta \frac{\delta^{(n+1)\beta}}{\epsilon^{\beta+1}}. \tag{26}$$

One can obtain them by estimating each term by (11) and (12) correspondingly, and then estimating the sum by an integral. The detailed proof of the first one can be found in [2, page 234], the second one can be proved in the same way.

In order to get the above estimate of $|T|$ and $|T'|$, we shall divide terms in the right-hand side of (19) into several groups according to their distance from the point t. For each group, we estimate $|f(c_Q)|$ by means of (21) (the closer to t the interval Q is, the better is this estimate) and then apply (25) (correspondingly, (26) for $|T'|$), which, by contrast, becomes better when ϵ grows.

So, let $G_{n+1}^g := \bigsqcup_{k \leq n+1} G^{[k]}$, where $G^{[n+1]} := G_{n+1}^g \backslash G_{\delta^n}$, $G^{[k]} := (G_{n+1}^g \cup G_{\delta^k}) \backslash G_{\delta^{k-1}}$. For $y \in G^{[k]}$ we have $|f_n(y)| \leq |f_n(t)|/\rho^{n-k+2}$ and $dist(G^{[k]}, t) \geq \delta^k/2$, therefore

$$|T_{n+1}(t)| \leq \sum_k \sum_{Q' \in G^{[k]}} |F_{Q'}^{[\varepsilon_n]}(t)||f_n(t)|/\rho^{n-k+2} \leq$$

$$\leq c(\alpha)\lambda^\beta |f_n(t)| \sum_k \frac{\delta^{(n+1)\beta}}{\delta^{k\beta}\rho^{n-k+2}} \leq c(\alpha)\lambda^\beta |f_n(t)| \rho^{-1} \sum_k \left(\frac{\delta^\beta}{\rho}\right)^{n-k+1};$$

similarly

$$|T'_{n+1}(t)| \le \sum_k \sum_{Q'\in G^{[k]}} |(F_{Q'}^{[\varepsilon_n]})'(t)||f_n(t)|/\rho^{n-k+2} \le$$

$$\le c(\alpha)\frac{\lambda^\beta |f_n(t)|}{\delta^{n+1}} \sum_k \frac{\delta^{(n+1)(\beta+1)}}{\delta^{k\beta+1}\rho^{n-k+2}} \le c(\alpha)\frac{\lambda^\beta |f_n(t)|}{\delta^{n+1}\rho} \sum_k \left(\frac{\delta^{\beta+1}}{\rho}\right)^{n-k+1}.$$

Taking δ according to the condition $\frac{\delta^\beta}{\rho} < \frac{1}{2}$, we get (20) and (22).

Now let us prove that (21) and (23) follow from (20), (22) and (24) for the previous n. We need an estimate

$$|f_n(x)| \le \frac{4}{\theta}|f_{n+1}(x)| \le \ldots \le \left(\frac{4}{\theta}\right)^k |f_{n+k}(x)|, \quad x \in I,\ n,k \in \mathbb{N}, \tag{27}$$

which holds for λ sufficiently small. Let us prove it. Let $Q_x^{n+1} \in G_{n+1}^g$. Then we have

$$|f_n(x)| \le \frac{2}{\theta}|f_n(x) - f_n(c_{Q_x})e^{i\theta}F_{Q_x}^{[\varepsilon_n]}(x)| \le$$

$$\le \frac{2}{\theta}(|f_{n+1}(x)| + |T_{n+1}(x)|) \le \frac{2}{\theta}(|f_{n+1}(x)| + c(\alpha)\lambda^\beta\rho^{-1}|f_n(x)|).$$

The first inequality follows from point 2 of Lemma 4 (which is applicable because of (24)), the last one – from (20). Now, the inequality $|f_n(x)| \le \frac{4}{\theta}|f_{n+1}(x)|$ follows from the last estimate, if $2c(\alpha)\lambda^\beta\theta^{-1}\rho^{-1} < \frac{1}{2}$. If $Q_x^{n+1} \notin G_{n+1}^g$, then it follows from (20) even easier. So, (27) is proved.

Now, let $k \le n+1$, $x \in V_{n+2}^g$, $y \in I$ and $|x-y| \le \delta^k$. Let $k' := \max\{l \le n : x \in V_{l+1}^g\}$. From the fact that x is again in a "good" set V_{n+2}^g, it follows that

$$x \in Q_x^{k'+1}[\tau^{n-k'+1}]. \tag{28}$$

First assume $k \le k'$. Then

$$|f_{n+1}(x)| = |f_n x| + |T_{n+1}(x)| \le |f_n(x)|(1 + c(\alpha)\frac{\lambda^\beta}{\rho}) \le \ldots \le$$

$$\le |f_{k'+1}(x)|(1 + c(\alpha)\frac{\lambda^\beta}{\rho})^{n-k'} \le$$

$$\le (|f_{k'}(x)| + |T_{k'+1}(x)| + |f_{k'}(c_{Q_x^{k'}})||F^{[\varepsilon_n]}_{Q_x^{k'+1}}|)(1 + c(\alpha)\frac{\lambda^\beta}{\rho})^{n-k'} \le$$

$$\leq |f_{k'}(x)|(1 + c(\alpha)\frac{\lambda^\beta}{\rho} + (1+\kappa)|F^{[\varepsilon_n]}_{Q_x^{k'+1}}|)(1 + c(\alpha)\frac{\lambda^\beta}{\rho})^{n-k'}.$$

Applying (13) and taking into account (28), we write on:

$$|f_{n+1}(x)| \leq |f_{k'}(x)|(1 + C(\alpha)\lambda^\beta(\frac{1}{\rho} + \frac{(1+\kappa)}{\tau^{(n-k'+1)\beta}}))(1 + c(\alpha)\frac{\lambda^\beta}{\rho})^{n-k'} \leq$$

$$\leq \frac{|f_{k'}(y)|}{\rho^{k'-k+1}}(1 + C(\alpha)\lambda^\beta(\frac{1}{\rho} + \frac{(1+\kappa)}{\tau^{(n-k'+1)\beta}}))(1 + c(\alpha)\frac{\lambda^\beta}{\rho})^{n-k'} \leq$$

$$\leq |f_{n+1}(y)|\frac{\left(\frac{4}{\theta}\right)^{n-k'+1}}{\rho^{k'-k+1}}(1 + C(\alpha)\lambda^\beta(\frac{1}{\rho} + \frac{(1+\kappa)}{\tau^{(n-k'+1)\beta}}))(1 + c(\alpha)\frac{\lambda^\beta}{\rho})^{n-k'}.$$

We have used the induction assumption (21) for $n = k'$ and (27). Now, if we choose ρ so that $\sqrt{\rho} < \theta/50$ and $\sqrt{\rho} < \tau^\beta/50(C(\alpha))$, we get

$$|f_{n+1}(x)| \leq |f_{n+1}(y)|\frac{1}{\rho^{k'-k+1}}5^{-n-k'+1}\rho^{-n+k'-1}\cdot$$

$$\cdot((1 + c(\alpha)\frac{\lambda^\beta}{\rho})^{n-k'+1} + 5^{-n-k'+1}\rho^{-n+k'-1}(1 + c(\alpha)\frac{\lambda^\beta}{\rho})^{n-k'}).$$

Then we choose $\lambda = \lambda(\rho)$ so that $(1 + c(\alpha)\frac{\lambda^\beta}{\rho}) < 2$, and get

$$|f_{n+1}(x)| \leq \frac{|f_{n+1}(y)|}{\rho^{n-k'+1+k'-k+1}},$$

and we are done.

Now we prove (23). Let $t \in V^g_{n+2}$. Let, as in the proof of (21), $k' := \max\{l \leq n : x \in V^g_{l+1}\}$. Again there holds (28). We have:

$$|(f_{n+1})'(t)| \leq |f'_n(t)| + |T'_{n+1}(t)| \leq \ldots \leq$$

$$\leq |f'_{k'}(t)| + |f_{k'}(c_{Q_t^{k'+1}})||(F^{[\varepsilon_{k'}]}_{Q_t^{k'+1}})'(t)| + \sum_{l=k'+1}^{n+1}|T'_l(t)|. \tag{29}$$

Applying the induction assumption ((23) for $n = k'$) and (27), we have

$$|f'_{k'}(t)| \leq c_1(\alpha)\lambda^\beta\frac{|f_{k'}(t)|}{\delta^{k'}} \leq c_1(\alpha)\lambda^\beta(4/\theta)^{n-k'+1}\frac{|f_{n+1}(t)|}{\delta^{k'}}.$$

We estimate the second term in (29) using (24), then (28) and (14) and finally (27), in the following way:

$$|f_{k'}(c_{Q_t^{k'+1}})||(F^{[\varepsilon_{k'}]}_{Q_t^{k'+1}})'(t)| \le (1+\kappa)|f_{k'}(t)|\frac{c(\alpha)\lambda^\beta}{\delta^{k'+1}\tau^{\beta(n-k'+1)}} \le$$

$$\le (1+\kappa)(4/\theta)^{n-k'+1}|f_{n+1}(t)|\frac{c(\alpha)\lambda^\beta}{\delta^{k'+1}\tau^{\beta(n-k'+1)}}.$$

Finally, using (22) and then again (27), we get

$$|T'_{l+1}(t)| \le \frac{c(\alpha)\lambda^\beta|f_l(t)|}{\rho\delta^{l+1}} \le (4/\theta)^{n+1-l}\frac{c(\alpha)\lambda^\beta|f_{n+1}(t)|}{\rho\delta^{l+1}}$$

Taking $\delta < \theta\tau^\beta/(4A)$ and substituting all the estimates into (29), we get

$$|f'_{n+1}(t)| \le \frac{\lambda^\beta|f_{n+1}(t)|}{\delta^{n+1}}(c_1(\alpha)/A^{n-k'+1} + c(\alpha)\frac{4(1+\kappa)}{\theta\tau^\beta} + \frac{4}{\rho\theta}\sum_{k=0}^{n-k'+1} A^{-k}).$$

Taking $A > 2$, we get (23), if $c_1(\alpha)$ is sufficiently large.

8. The end of the proof. In order to complete the construction, we should define the sets G_n. Fix a parameter τ in order to satisfy inequality $C(\tau)\sum_{n=1}^\infty \frac{1}{n^2} < \frac{1}{8}$, where $C(\tau)$ is a constant from Lemma 5.

We define the sets G_{n+1} as follows: $G_1 := H_1 = \{I\}$; an interval $Q \in H_{n+1}$ belongs to G_{n+1}, if $Q \subset V_n$,

$$M_Q(f_n) \le K_n\eta^n, \tag{30}$$

where K_n and η will be defined later and, besides,

$$\widetilde{D_n}(c_Q) \le n/2. \tag{31}$$

(in fact, of course, $\widetilde{D_n}$ is a constant on Q). Note that the choice of τ and Lemma 5 guarantee that the total (for all n's) length of $Q \in H_{n+1}$, $Q \subset V_n$, not included in G_{n+1} because of violation of the condition (31), does not exceed $\frac{1}{8}$.

Lemma 7 *There exists a constant $C'(\alpha)$ such that for all sufficiently small λ the following inequalities hold:*

1) If $Q \in G^g_{n+1}$, then

$$\int_Q |f_{n+1}|^p \le X \int_Q |f_n|^p.$$

2) If $Q \in G^d_{n+1}$, then

$$\int_Q |f_{n+1}|^p \le Y \int_Q |f_n|^p,$$

where $X := \gamma(\lambda)^p(1 + C'(\alpha)\lambda^\beta)^p (= (1 + B\lambda)(1 + C'(\alpha)\lambda^\beta)^p)$, $Y := (1 + C'(\alpha)\lambda^\beta)^p$

Proof. Letting $P_{n+1}(t) := f_n(t) - f_n(c_{Q_t})e^{i\theta}F^{[\varepsilon]}_{Q_t}(t)$, we get for $Q \in G^g_{n+1}$:

$$\int_Q |f_{n+1}|^p = \int_Q |P_{n+1} + T_{n+1}|^p \le \int_Q |P_{n+1}|^p(1 + \frac{|T_{n+1}|}{|P_{n+1}|})^p.$$

Applying point 2 of Lemma 4 and the estimate (20), we get

$$\int_Q |f_{n+1}|^p \le \int_Q |P_{n+1}|^p(1 + \frac{2c(\alpha)\lambda^\beta}{\rho\theta})^p,$$

and then, estimating the integral using point 1 of Lemma 4, we have

$$\int_Q |f_{n+1}|^p \le (1 + B\lambda)(1 + C'(\alpha)\lambda^\beta)^p \int_Q |f_n|^p.$$

Recall that applicability of Lemma 4 is provided by (24).

The second case is even easier.

Lemma 8 *If λ is sufficiently small, then for all n there holds an inequality*

$$\int_{V_n} |f_n|^p \le \eta^n \int_{V_1} |f_1|^p \tag{32}$$

with some $\eta \in (0, 1)$.

Proof. Denote as Θ_n the set $\{0, 1\}^n$ (the set of all ordered sets $v = (v_1, \dots, v_n)$ of the length n of zeroes and unities). Define for $v \in \Theta_n$ the set $G^{(v)} \subset G_n$ as the set of all Q's such that for all k $v_k = 1$ if and only if

$Q \subset V_k^g$. Let for $v \in \Theta_n$ $Z(v) := \prod\limits_{k=1}^{n} X^{v_k} Y^{(1-v_k)}$. Let by definition $G_1^g := I$ and prove by induction in n the estimate

$$\int_I |f_1|^p \geq X \sum_{v \in \Theta_n} Z(v)^{-1} \int_{G^{(v)}} |f_n|^p. \tag{33}$$

The base is obvious. Suppose (33) holds for some n. Then

$$Z(v)^{-1} \int_{G^{(v)}} |f_n|^p = Z(v)^{-1}\left(\int_{G^{(v,1)}} |f_n|^p + \int_{G^{(v,0)}} |f_n|^p\right) \geq$$

$$\geq Z(v)^{-1}\left(X^{-1} \int_{G^{(v,1)}} |f_{n+1}|^p + Y^{-1} \int_{G^{(v,0)}} |f_{n+1}|^p\right) =$$

$$= Z(v,1)^{-1} \int_{G^{(v,1)}} |f_{n+1}|^p + Z(v,0)^{-1} \int_{G^{(v,0)}} |f_{n+1}|^p.$$

(we used Lemma 7). Thus we have proved the inductive step. Then, for λ sufficiently small we have $X < 1$, $Y > 1$. If $n > N_0$, $v \in \Theta_n$, then $\sum_k v_k < n/2$ implies $G^{(v)} = \varnothing$ (it follows from (31)), therefore the right-hand side in (33) is not less than $X^{-\frac{n}{2}+1} Y^{-\frac{n}{2}} \int_{V_n} |f_n|^p$, and (32) follows provided λ is sufficiently small.

Now we are ready to finish the setup of the construction. What we should do is to make precise the condition (30). We take η from Lemma 8. As, by Lemma 8, $\int_{V_n} |f_n|^p \leq C\eta^n$, the total length of all intervals $Q \in H_{n+1}$, such that $Q \subset V_n$ and on Q the condition (30) fails, does not exceed $\frac{C}{K_n}$. For K_n we take a sequence growing as some power of n and satisfying the condition $\sum_{n=1}^{\infty} \frac{C}{K_n} < \frac{1}{8}$. We get that total length of all intervals dropped out of V_n on all steps because of violation of (30) is less than $\frac{1}{8}$. But one can tell the same thing about the length of all intervals dropped out because of violation of (31). Hence $|\bigcap_{i=1}^{\infty} V_n| > \frac{3}{4}$.

It follows from (30) and an estimate (24) of the oscillation of f_n that if $Q \in G_{n+1}$, then $|f_n(c_Q)| \leq C'\eta'^n$. Hence, taking into account (3) and (25) we get

$$|f_n(t) - f_{n+1}(t)| \leq \frac{C_1 \eta'^n}{\varepsilon_n^\beta} + C_2 \lambda^\beta \eta'^n, t \in \mathbb{R} \tag{34}$$

The first term in the right-hand side corresponds to the building block $F_{Q_t^{n+1}}^{[\varepsilon_n]}$ (if there is any), the second – to all the others. It follows from this estimate

that, f_n converges uniformly on $\mathbb{R}$ to some function f. We also need an estimate

$$|f_n(t) - f_{n+1}(t)| \le c\frac{\lambda^\beta \delta_{n+1}^\alpha C' \eta'^n}{|t|^\beta}, \qquad t \notin 3I, \tag{35}$$

which follows from (11). Now, (35) implies that

$$|f_n(t)| \le c|t|^{-\beta}, \quad |t| \ge 3/2, \tag{36}$$

so we can, fixing t, write

$$\int\limits_{\mathbb{R}} \frac{f_n(s)ds}{|s-t|^{1-\alpha}} = \int\limits_{|s| \le \max(2|t|, 3/2)} \frac{f_n(s)ds}{|s-t|^{1-\alpha}} + \int\limits_{|s| \ge \max(2|t|, 3/2)} \frac{f_n(s)ds}{|s-t|^{1-\alpha}}.$$

In the first term, the passage to the limit in the integral is provided by uniform convergence of f_n as $n \to \infty$, in the second one, the function under integral is majorized by $c|s|^{-2}$. Hence

$$\lim_{n\to\infty} g_n = \lim_{n\to\infty} \int\limits_{\mathbb{R}} \frac{f_n(s)}{|s-t|^{1-\alpha}} = \int\limits_{\mathbb{R}} \frac{f(s)}{|s-t|^{1-\alpha}}$$

(The first inequality follows from Lemma 1). Finally note that the functions g_n converge uniformly to g:

$$|r_n| \le \frac{C'' \eta'^n}{\varepsilon_n}.$$

So, the program announced in the beginning of Section 3 is completed. Let us now show that the function f satisfies Hölder's condition.

In this argument, constants may depend on all parameters except n. Of course, we use the proposition from the beginning of Section 4. The condition $|f_n - f| \le C_1 \eta_1^n$ follows easily from (34).

Let us estimate the derivative of the correcting term on the n-th step:

$$|(W_\alpha r_n)'(t)| = |\frac{d}{dt}(\sum_{Q \in G_{n+1}^g} f_n(c_Q) F_Q^{[\varepsilon_n]}(t) e^{i\theta})| \le \max_{Q \in G_{n+1}^g} |f_n(c_Q)| \sum_{Q \in G_{n+1}^g} |(F_Q^{[\varepsilon_n]}(t))'| \le$$

$$\le \max_{Q \in G_{n+1}^g} |f_n(c_Q)| \left(|(F_{Q_t^{n+1}}^{[\varepsilon_n]}(t))'| + \sum_{Q \in H_{n+1},\, Q \ne Q_t^{n+1}} |(F_Q^{[\varepsilon_n]}(t))'| \right).$$

We know that $\max_{Q\in G^g_{n+1}} |f_n(c_Q)| \le C'\eta'^n$. Then, we use (8) for the first term in the brackets, and for the sum – (26) with $\epsilon := \frac{\delta^{n+1}}{2}$. Thus we have

$$|(W_\alpha r_n)'(t)| \le C'\eta'^n(\frac{C_1}{\delta^{n+1}\varepsilon_n} + \frac{C_2}{\delta^{n+1}}) \le \frac{C_3}{\delta'^{n+1}}$$

Hence $|f'_n(t)| \le \frac{C_3}{1-\delta'}\delta'^{-n-1} \le \frac{C_4}{\delta'^{n+1}}$, and Hölder's property for f is proved.

9. Remark on the order of choice of the parameters. Recall the order we chose our parameters in. Given $\alpha \in (0,1)$, we fix p and θ (Lemmas 3 and 4). Here Lemma 4 holds for all sufficiently small λ and ε. Then we fix a sequence ε_n. Independently of other parameters we fix τ. Lemma 6 holds for all sufficiently small λ and δ, independently of the choice of G_n (here how small λ and δ should depend on ρ from this lemma, and ρ itself only depends on α, θ, p, τ, and the setup function g_1). Now we choose λ such that the multiplier η in front of the integral in the right-hand side of (32) is less than 1 (decreasing λ killing the influence of "tails" T_n in comparison with the correction effect provided by point 1 of Lemma 2. Note that the last effect decays when λ decreases as well ($\gamma(\lambda) \xrightarrow{\lambda\to 0} 1$!), but the "tails" dies faster). Finally, we fix δ in order to provide (24).

So far our considerations did not depend on G_n, therefore we did not need to define them. Now we fix a sequence K_n and this finishes the definition of our construction.

10. The case of negative α. One can ask whether there is an analogue of Theorem 1 for other M.Rietz's kernels. In the case of the kernel $|x|^{-\beta}, \quad 1 < \beta < 2$, the answer is affirmative; moreover, as the convolution with such a kernel is, at least formally, the inverse operator to $U_{\beta-1}$ (see Lemma 1, points 2 and 5), the example in essential coincides with the one built above.

Theorem 2 *There exist a nonzero continuous function $g : \mathbb{R} \to \mathbb{C}$, $\operatorname{supp} g \subseteq 3I$ and a set E of positive measure, such that $t \in E \Rightarrow g(t) = 0$, $\int_{\mathbb{R}} g(x)|t-x|^{-\beta}dx = 0$. The last integral converges absolutely for every $t \in E$. The function g satisfies Hölder's condition with an exponent $\beta - 1$.*

Proof. We take for g the function built in Theorem 1 (with $\beta - 1$ for α). Let

$$E := V \cap (I\backslash S), \quad \text{where } S := \bigcup_{n\in\mathbb{N}} \bigcup_{Q\in G^g_{n+1}} 3\varepsilon_n(Q - c_Q) + c_Q$$

The idea is that $|S| < \frac{3}{4}$, but now *supp* g_n is contained "rather deep" inside S:

$$dist(supp\ g_n, E) \geq \varepsilon_n \delta^{n+1}. \tag{37}$$

We know (point 5 of Lemma 1), that for $t \in E$

$$f_n(t) = \int_{\mathbb{R}} g_n(x)|t-x|^{-\beta} dx \overset{n\to\infty}{\longrightarrow} 0.$$

To prove Theorem 2, it is sufficient to justify the passage to the limit in the integral. For the summable majorant we take a function

$$\widetilde{g}(x) := |t-x|^{-\beta} \sum_{n=1}^{\infty} |g_{n+1}(x) - g_n(x)|.$$

Let us estimate the n-th term:

$$|g_{n+1} - g_n| \leq \sum_{Q \in G^g_{n+1}} |f_n(c_Q)|(\delta^{n+1}\lambda)^{\alpha}(\phi_{\varepsilon_n Q}) \leq C'\eta'^n(\delta^{n+1}\lambda)^{\alpha} \sum_{Q \in G^g_{n+1}} (\phi_{\varepsilon_n Q}).$$

Check summability of the majorant:

$$\int_{\mathbb{R}} |t-x|^{-\beta}|g_{n+1}(x) - g_n(x)| \leq C'\eta'^n(\delta^{n+1}\lambda)^{\alpha} \sum_{Q \in G^g_n} \int_{\mathbb{R}} |t-x|^{-\beta}\phi_{\varepsilon_n Q}(x)dx \leq$$

$$\leq C'\eta'^n(\delta^{n+1}\lambda)^{\alpha} \sum_{Q \in G^g_n} |t-x^*_Q|^{-\beta} \int_{\mathbb{R}} \phi_{\varepsilon_n Q}(x)dx \leq$$

$$\leq C'\eta'^n(\delta^{n+1}\lambda)^{\beta} \sum_{Q \in G^g_n} |t-x^*_Q|^{-\beta}. \tag{38}$$

Here x^*_Q denotes a point of the support of $\phi_{\varepsilon_n Q}(x)$, closest to t. Using (37) and the fact that the distance between two different points x^*_Q and $x^*_{Q'}$ is no less than δ^{n+1}, we get

$$\sum_{Q \in G^g_n} |t-x^*_Q|^{-\beta} \leq 2\sum_{k=0}^{\infty}(\varepsilon_n\delta^{n+1} + k\delta^{n+1})^{-\beta} \leq 2\delta^{(n+1)(-\beta)} \sum_{k=0}^{\infty} |\varepsilon_n + k|^{-\beta} \leq$$

$$\leq 2\delta^{(n+1)(-\beta)}(\varepsilon_n^{-\beta} + \sum_{k=1}^{\infty} k^{-\beta}) \leq C\delta^{(n+1)(-\beta)}\varepsilon_n^{-\beta} \tag{39}$$

Substituting (39) to (38) and summing over all n, we get

$$\int_{\mathbb{R}} \widetilde{g}(t)dt \leq C''\lambda^{-\beta} \sum_{n=1}^{\infty} \eta'^n \varepsilon_n^{-\beta} < +\infty$$

We should now only prove that g satisfies Hölder's condition with an exponent $\alpha = \beta - 1$. In fact this is a property of the potential U_α of any bounded function for which it is defined. To prove it, take $t > 0$ and write the following estimate:

$$\int_{\mathbb{R}} ||x|^{\alpha-1} - |x-t|^{\alpha-1}|dx = \int_{(-t;2t)} + \int_{\mathbb{R}\setminus(-t;2t)} =: J_1 + J_2.$$

Estimate each term:

$$J_1 \leq \int_{(-t;2t)} |x|^{\alpha-1} + \int_{(-t;2t)} |x-t|^{\alpha-1} = \frac{2}{\alpha}(1+2^\alpha)t^\alpha;$$

$$J_2 \leq 2\int_{(t;+\infty)} x^{\alpha-1} - (x+t)^{\alpha-1} \leq 2(\alpha-1)\int_{(t;+\infty)} tx^{\alpha-2} \leq 2t^\alpha.$$

When estimating J_2 we have used the inequality $|h(x+t)-h(x)| \leq t \sup\limits_{s\in(x,x+t)} |h'(s)|$ for a smooth function h. From this estimates we get

$$|(U_\alpha f)(t+\delta)-(U_\alpha f)(t)| \leq \sup_{\mathbb{R}} |f| \int_{\mathbb{R}} (|t+\delta-x|^{\alpha-1} - ||t-x|^{\alpha-1}|)dx \leq C(\alpha)(\sup_{\mathbb{R}} |f|)\delta^\alpha.$$

The theorem is proven.

Remark 1. Proving the smoothness of g we didn't use all the information we had about f. In fact f, besides it is bounded and belongs to the domain of U_α, satisfies Hölder's condition with some exponent $r > 0$. Using this and well-known techniques of estimating operators similar to M. Rietz potential (see, for example, [8]), one can prove that g satisfies Hölder's condition with an exponent $\beta - 1 + r$.

Remark 2. The theorem proven in [5] states that for the potentials U_α, $-1 < \alpha < 0$, uniqueness holds if the density g belongs to $C^{1+\varepsilon}$ with some $\varepsilon > 0$. Theorem 2 shows that the last condition cannot be replaced by Hölder's condition with an exponent $-\alpha$. So, there is a gap between the two results, which decreases when $\alpha \to -1$. If $\alpha \leq -1$, one does not need any supplementary smoothness condition: the mere existence of the potential is sufficient (see, for example, [6]).

For the cases $\alpha = 0$ and $\alpha > 1$ (except odd integers for which the uniqueness does not hold in any sense), the question whether one can omit the smoothness conditions imposed in [5] remains open.

11. Extension of the results to the multidimensional case. The result of Theorem 1 can be extended to the case of M. Rietz potentials in spaces $\mathbb{R}^d$ for $d > 1$. In this case for $\alpha \in (0, d)$ we consider a set of all measurable functions f, satisfying the condition

$$\int_{\mathbb{R}^d} \frac{|f(x)|}{1 + |x|^{d-\alpha}} dx < +\infty \tag{40}$$

(as above, we denote this set $dom\, U_\alpha$). We let

$$U_\alpha f := f * |x|^{d-\alpha},\ f \in dom\, U_\alpha,$$

where $*$ denotes the convolution in $\mathbb{R}^d$. The case of major interest is $d = 2$; $\alpha = 1$ (Newton's potential of the charge concentrated in the plane). Note, however, that in case of $d > 1$ there is not any analogue of the uniqueness theorem mentioned in the introduction.

There holds the following generalization of Theorem 1:

Theorem 3 *For all $d \in \mathbb{N}$ and for all $\alpha \in (0, d)$ there exist a nonzero function $f : \mathbb{R}^d \to \mathbb{R}$, $f \in dom\, U_\alpha$ and a set $E \subset \mathbb{R}^n$ of positive Lebesgue measure satisfying the condition $f|_E = 0$, $U_\alpha f|_E = 0$, and Hölder's condition with some positive exponent.*

The proof of this theorem is quite similar to the one of Theorem 1. We highlight some details differing in the multidimensional case.

We need an operator W_α, "the inverse operator" to U_α. The precise expression of this operator (see, for example, [2, page 241]) does not matter for us, the only thing we need is that if we now denote as β the number $d+\alpha$, then points 1,2,3,5 of Lemma 1 still hold.

The role of I will be played by the cube I^d, and we shall consequently divide it to congruent cubes with the side equal to δ^n. Instead of "the finitizator" $\phi(x)$ we take the function $\phi(|x|)$

The computations made in [2, page 241] show that lemma 2 still holds in the multidimensional case. Lemmas 3 and 4 can be derived from it quite similarly to the above.

Taking into account that now $\beta = d + \alpha$, the most of computations in the multidimensional case will repeat one-dimensional literally, if we also replace derivative by gradient everywhere. So, because of point 3 for lemma 1 there still hold (3), (11), (12), (13), (14) (now for the cube Q with sides parallel to the coordinate axes the symbol $Q[a]$ denotes $Q \backslash Q'$, where Q' is a cube obtained from Q by homothety with the center c_Q and the dilation factor a).

The remaining part of the construction is the same. Note that estimates (25) and (26), playing the key role in the proof of lemma 6, and seeming to depend on the dimension, in fact hold in the literally same form.

Remark. Theorem 2 can be extended to the multidimensional case as well: *for $d < \beta < 2d$ there exist a nonzero continuous function $g : \mathbb{R}^d \to \mathbb{C}$ and a set E of positive measure, such that if $t \in E$, then $g(t) = 0$, $\int_{\mathbb{R}^d} g(x)|t-x|^{-\beta}dx = 0$. The last integral converges absolutely for all $t \in E$. The function g satisfies Hölder's condition with an exponent* $\min\{\beta - d; 1\}$[5]. The only difference in the proof is the estimate (39), where sums becomes multiple (of order d). They still will converge, because now $\beta > d$.

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